Did A Jormungand State Exist?

An investigation using the Budyko-Widiasih model

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Outline

Our investigation will be conducted as follows:

1. Begin by looking at the scientific data.
2. Introduce the Budyko-Widiasih model and its conclusions.
3. Examine the results of numerical simulations to the Budyko-Widiasih model.
4. Find an approximation to solutions to the Budyko-Widiasih model.

The Possibilities

- Geological and paleomagnetic evidence indicates that glaciers grew near the equator during the last two Neoproterozoic glacial periods.
- There are different hypotheses as to the exact nature of these glaciations:

1. The Snowball Earth Hypothesis: Glaciers covered the entirety of the Earth's surface.
2. The Slushball Earth Hypothesis: Continents completely covered in ice, belt of free ocean.
3. The Thin-Ice Hypothesis: Glaciers completely covered the Earth, but the ice is thin at the tropics.
4. The Jormungand Hypothesis: Glaciers mostly covered the Earth, ice is not snow covered near the tropics, and a belt of free ocean existed.

Evidence for Extreme Glaciations

- The magnetic orientations of rocks tell us the continents were near the equator 750 million and 580 million years ago.
- But there was glacial debris on these continents during this period:
  - Glaciers today only survive 5,000 meters above sea level (4,000 in the last ice age).
  - These contained iron-rich rocks which implies little to no oxygen in the oceans and atmosphere.
  - Rocks known to form in warm water accumulated just after the glaciers receded (evidence for strong hysteresis).
- Just after the proposed glaciation is the Cambrian Explosion.

Reference: *Snowball Earth*, Scientific American, Hoffman and Schrag.

The Snowball Earth Hypothesis

- Glaciers covered the entirety of the Earth's surface.
- Life survived in small communities near hot springs.
- The isolation explains the Cambrian Explosion.
- CO_2 built up because of the lack of silicate weathering caused the abrupt change.

Evidence Against the Snowball Earth Hypothesis

- There is evidence that many sponges survived the Neoproterozoic glaciations.
- There is evidence that photosynthetic eukaryotes thrived both before and immediate after the Snowball episodes.
- New evidence that life can survive under miles of glaciers does not apply to complex life.

Reference: *The Jormungand Glocal Climate State and Implications for Neoproterozoic Glaciations*, Abbot et al.

The Slushball Earth Hypothesis

- Continents completely covered in ice, belt of free ocean.
- This would allow complex life and photosynthetic eukaryotes survive.
- The Slushball models do not seem to have a strong enough hysteresis to account for the CO_2 measurements.
  - The Slushball Model of Liu and Peltier (2010) occur with CO_2 $\mathcal{O}(100-1000)$ ppmv
  - Measurements by Bao et al. indicate values one or two orders of magnitude more!

The Thin-Ice Hypothesis

- Glaciers completely covered the Earth, but the ice is thin at the tropics.
- The thin ice would allow photosynthetically active radiation to penetrate to the ocean below.
- Such a solution is possible if bare sea ice has a high transmissivity and an albedo lower than that of snow covered ice.
- Has not been found in a global climate model so far, and there is debate as to whether the parameter regime is realistic

The Jormungand Hypothesis

- Mixture of Slushball Earth and Thin-Ice.
- Ice sheets almost cover the entire Earth, though those near the tropics are not covered in snow.
- This is a solution that global climate models have found.
- "There is strong hysteresis associated with the Jormungand state, which is to say that the Jormungand state and one or both of the other states are stable for a wide range of pCO2". (Abbot 2011)

Introduction to the Budyko-Widiasih Model

- The Budyko-Widiasih Model is an Energy Balance Model (EBM).
- It is designed to examine the movement of the ice-line.
- As a lower order model, dynamical systems theory can be used to verify the existence of a Jormungand state.

The Budyko-Widiasih Model

- y is the sine of the compliment of the polar angle (the latitude as written from 0 to 1).
- η is the ice-line, the latitude of furthest extent of the Earth's polar glaciers.
- $T(y,\eta)$ is the annual average surface temperature as a function of latitude and the ice-line.
- M is the meridional heat transport, the transport of heat from one latitude to another.
- These quantities are related in the following manner:

$$\begin{aligned} R\frac{\partial T}{\partial t} &= E_{in} - E_{out} - M, \\ \frac{\partial \eta}{\partial t} &= \epsilon(T(\eta,\eta) - T_c). \end{aligned}$$

- $T(\eta,\eta) = \frac{1}{2}(\lim_{y\to\eta^-} T(\eta,\eta) + \lim_{y\to\eta^+} T(\eta,\eta))$.
- T_c is the critical temperature to melt the glaciers, $-10^\circ C$.

The Energy Terms

$$E_{in} = Qs(y)(1-\alpha(y,\eta)), \quad E_{out} = A + BT, \quad M = C(T-\overline{T}).$$

- s describes the distribution of the insolation as a function of latitude. It can be well-approximated by a quadratic.
- α is the albedo of a latitude as a function of the ice-line.
- $\overline{T} = \int_0^1 T(y)dy$, the average temperature of the Earth's surface.
- E_{out} is the Budyko-Sellers model verified by satellite data.
- M is a relaxation to the mean.

The Albedo Function

The Budyko model used the albedo function as follows:

$$\alpha(y,\eta) = \begin{cases} \alpha_s, & y > \eta \\ \alpha_w, & y < \eta \\ \frac{1}{2}(\alpha_s + \alpha_w), & y = \eta, \end{cases}$$

where $\alpha_s > \alpha_w$.

Solution to the Budyko-Widiasih Model

We are looking for a solution to understand the dynamics of the ice-line, that is a function h that satisfies

$$\frac{\partial \eta}{\partial t} = \epsilon h(\eta).$$

Solutions to the Budyko-Widiasih Model satisfy

$$h(\eta) = \frac{Q}{B+C}\left(s(\eta)(1-\alpha(\eta,\eta)) + \frac{C}{B}(1-\overline{\alpha}(\eta))\right) - \frac{A}{B} - T_c,$$

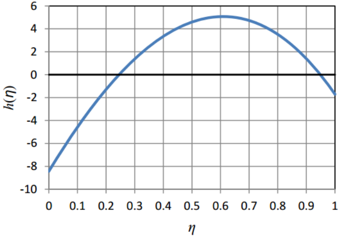
where

$$\overline{\alpha}(\eta) = \int_0^1 \alpha(y,\eta)s(y)dy.$$

McGehee-Widiasih

McGehee and Widiasih utilized the Budyko albedo function and solved for an approximation to

$$\frac{\partial \eta}{\partial t} = \epsilon h(\eta).$$



An Extension to the Albedo Function

- We wish to introduce the idea of bare sea ice into the albedo function.
- As noted before, the albedo of bare sea ice is less than that of snow covered ice.
- When the ice-line grows past a certain latitude ρ, it enters the Hadley cell circulation zone.
- This would lead to more evaporation than precipitation leading to bare sea ice.

Abbot's Albedo Function

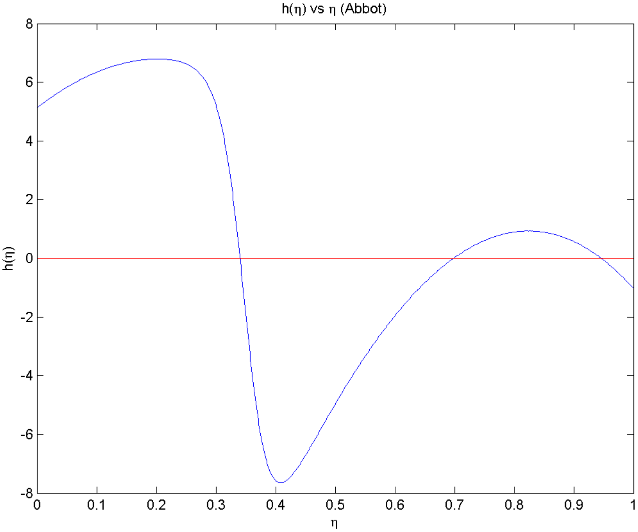
Abbot et al. introduced the idea using the following albedo function:

$$\alpha(y,\eta) = \begin{cases} \alpha_2(y), & y > \eta \\ \frac{1}{2}(\alpha_w + \alpha_2(\eta)), & y = \eta \\ \alpha_w, & y < \eta, \end{cases}$$

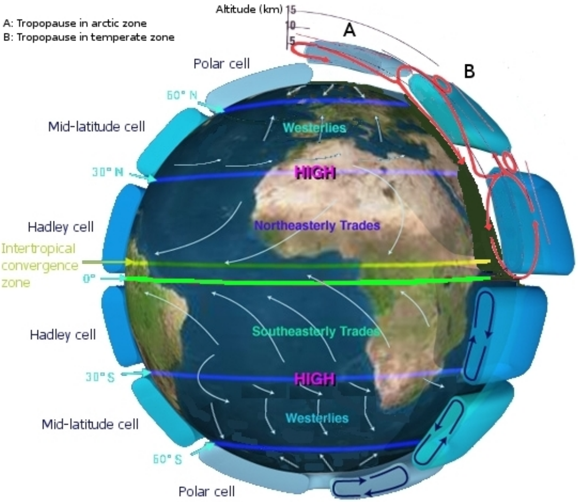
where

$$\alpha_2(\eta) = \frac{1}{2}(\alpha_s + \alpha_i) + \frac{1}{2}(\alpha_s - \alpha_i)\tanh(y-\rho).$$

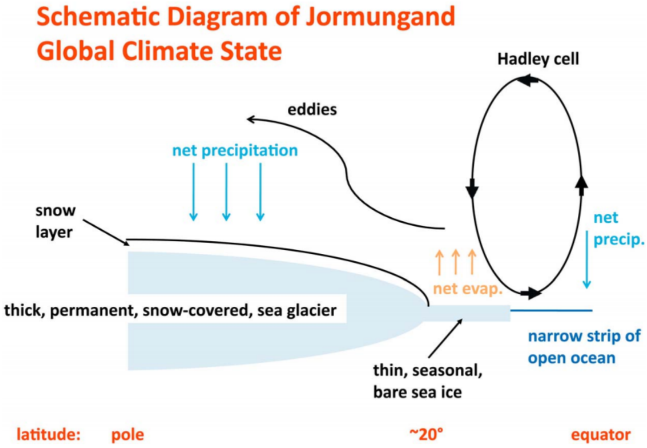
Numerical Solution Using Abbot's Albedo Function



Hadley Cells



Hadley Cell Effect



Hadley Cell Effect (Continued)

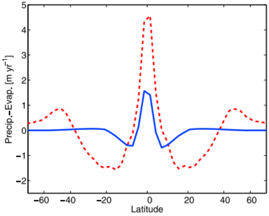


Figure 4. Annual and zonal mean precipitation minus evaporation for the ice-free state (red dashed) and the Jormungand state (blue) with pCO_2 = 5000 ppm.

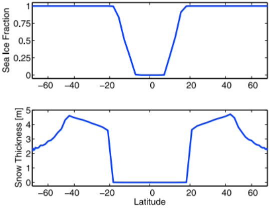
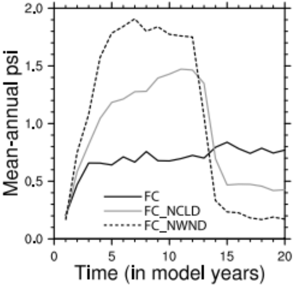


Figure 6. Annual and zonal mean (top) sea-ice fraction and (bottom) snow thickness in the Jormungand state (blue) with pCO_2 = 5000 ppm.

Hadley Cell Intensification

- Poulsen and Jacob examined the Hadley cells' circulation at the onset of Snowball Earth



- They concluded that the Hadley cell circulation abruptly intensifies and then abruptly weakens.

Resulting Albedo Function

- The resulting albedo function would then be complex:
  - As the ice-line heads towards the equator, the ice in the Hadley cell area would be mostly bare due to the evaporation.
  - How much the Hadley cell effect increases effects how close to α_i the albedo becomes.
- We can then understand the system by bounding its possibilities between two models:
  - An albedo function which becomes instantly α_i in the Hadley cell zone due to increased Hadley cell effect.
  - The albedo in the Hadley cell zone changes linearly from α_s to α_i.
- The albedo of the Earth's system will be underestimated in the first function, resulting in a maximum equilibrium ice-line.

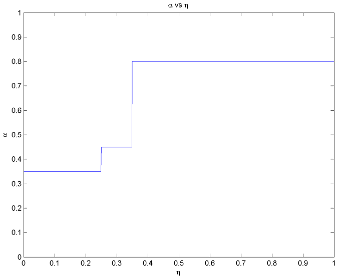
The Instant Jormungand Albedo Function

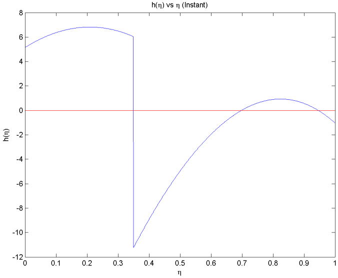
The Instant Jormungand Albedo Function is defined as:

$\eta < \rho$	$\eta > \rho$
$\alpha(y,\eta) = \begin{cases} \alpha_s, & y > \rho \\ \alpha_i, & \eta < y < \rho \\ \alpha_w, & y < \eta \\ \frac{1}{2}(\alpha_s + \alpha_b(\eta)), & y = \rho \\ \frac{1}{2}(\alpha_i + \alpha_w), & y = \eta \end{cases}$	$\alpha(y,\eta) = \begin{cases} \alpha_s, & y > \eta \\ \alpha_w, & y < \eta \\ \frac{1}{2}(\alpha_s + \alpha_w), & y = \eta \end{cases}$

where $\alpha_w < \alpha_i < \alpha_s$.

Instant Jormungand Albedo Function Solution





The Linear Jormungand Albedo Function

The Linear Jormungand albedo function is defined as:

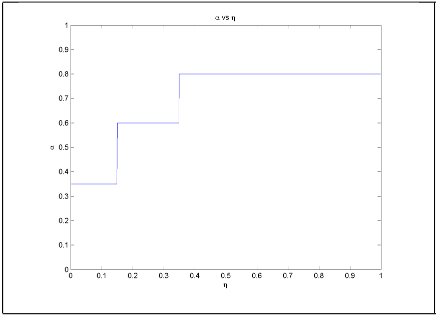
$\eta < \rho$	$\eta > \rho$
$\alpha(y,\eta) = \begin{cases} \alpha_s, & y > \rho \\ \alpha_b(\eta), & \eta < y < \rho \\ \alpha_w, & y < \eta \\ \frac{1}{2}(\alpha_s + \alpha_b(\eta)), & y = \rho \\ \frac{1}{2}(\alpha_b(\eta) + \alpha_w), & y = \eta \end{cases}$	$\alpha(y,\eta) = \begin{cases} \alpha_s, & y > \eta \\ \alpha_w, & y < \eta \\ \frac{1}{2}(\alpha_s + \alpha_w), & y = \eta \end{cases}$

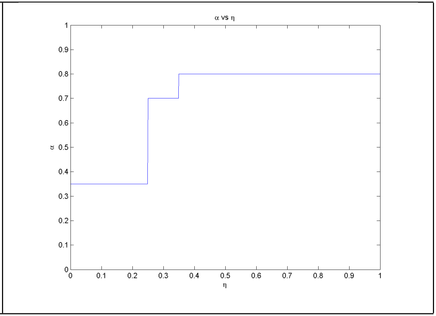
where

$$\alpha_b(\eta) = \frac{\alpha_s - \alpha_i}{\rho}\eta + \alpha_i,$$

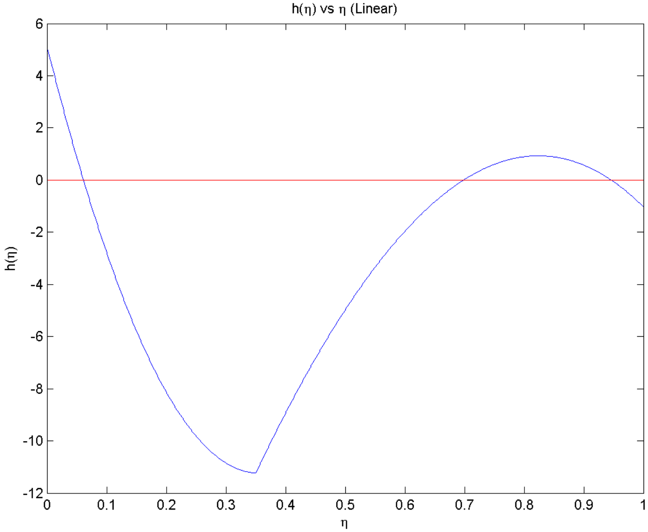
and $\alpha_w < \alpha_i < \alpha_s$.

Linear Jormungand Albedo Function Graphs





Linear Jormungand Albedo Function Solution



Stable Equilibrium Result

- The Budyko-Widiasih suggests that when taking into account the change due to the weakening (and eventual halt) of the Hadley cells, the stable large ice-line solution is a Jormungand state.
- This runs counter to the thin-ice and Snowball Earth hypotheses.
- The bifurcation diagram shows that the Jormungand model can produce the necessary hysteresis.

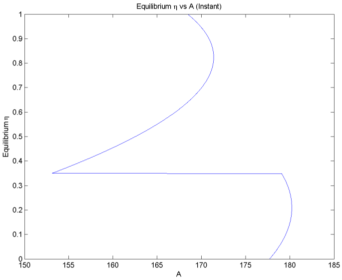
Hysteresis of the Jormungand Model

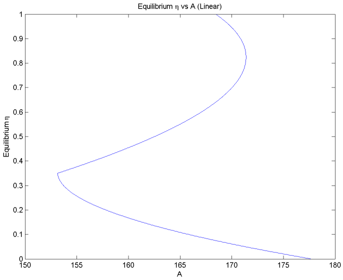
- We model the effect of increasing CO_2 levels as decreasing the term A.
- We can solve for the value of A required for an equilibrium η:

$$A(\eta) = \frac{B}{B+C}\left(Qs(\eta)1 - \alpha(\eta,\eta)) + \frac{C}{B}Q(1-\overline{\alpha}(\eta))\right) - BT_c.$$

- We can then use a bifurcation diagram to analyze the effect on the stable ice-line solutions.

Bifurcation Diagrams





Problem Statement

- We wish to find an analytical approximation to $h(\eta)$ for the Jormungand Linear model which does not require a numerical integration.
- We will begin following McGehee-Widiasih and end using a fast-slow approximation.
- Note: McGehee-Widiasih have already solve $h(\eta)$ for $\eta > \rho$, so our goal is to solve $h(\eta)$ piecewise.

Step 1: Split the Temperature Function

Let

$$T(y,t) = \begin{cases} U(y,t), & y < \eta \\ V(y,t), & \eta < y < \rho \\ W(y,t), & y \geq \rho \\ \frac{1}{2}(U(\eta,t) + V(\eta,t), & y = \eta. \end{cases}$$

Step 2: Second-Order Legendre Approximation

Assume

$$\begin{aligned} U(y,t) &= u_0(t)p_0(y) + u_2(t)p_2(y) \\ V(y,t) &= v_0(t)p_0(y) + v_2(t)p_2(y) \\ W(y,t) &= w_0(t)p_0(y) + w_2(t)p_2(y), \end{aligned}$$

where $p_0(y) = 1$ and $p_2(y) = \frac{1}{2}(3y^2 - 1)$.

Step 3: Write the Model in Terms of the u's, v's, and w's

$$
\begin{aligned}
\dot{\eta} &= \epsilon(T(\eta,\eta) - T_c) \\
\dot{u_0} &= \frac{1}{R}(Q(1-\alpha_w) - A - (B+C)u_0 + C\overline{T}(\eta)) \\
&\vdots \\
\dot{w_2} &= \frac{1}{R}(Qs_2(1-\alpha_s) - (B+C)w_2,
\end{aligned}
$$

where

$$
\begin{aligned}
T(\eta,\eta) &= \frac{1}{2}(u_0+v_0) + \frac{1}{2}(u_2+v_2)p_2(\eta), \\
\overline{T}(\eta) &= \eta u_0 - (\eta-\rho)v_0 + \frac{1}{2}(\eta^3-\eta)u_2 - (\frac{1}{2}(\eta^3-\eta) - k)v_2 + (1-\rho)w_0 - kw_2, \\
k &= \frac{1}{2}(\rho^3-\rho).
\end{aligned}
$$

Substitutions

Repeatedly substitute in functions of the u's, v's, and w's to eliminate variables and receive variables with solutions.
Many variables could then be written like:

$$\dot{e} = \frac{1}{R}((2Q(\alpha_s - \alpha_w) - (B + C)e)$$

which collapse over time to a single value. Thus our system becomes

$$\begin{aligned} \dot{\eta} &= \epsilon(T(\eta, \eta) - T_c), \\ \dot{a} &= \frac{1}{R}Q(1 - \frac{1}{2}(\alpha_s + \frac{1}{2}(\alpha_w + \alpha_i(\eta)))) - A - (B + C)a + C\overline{T}, \\ \dot{z} &= \frac{1}{R}(Q(\alpha_i(\eta) - \alpha_w) - (B + C)z), \end{aligned}$$

Fast-Slow Approximation

- McGehee-Widiasih estimates $\epsilon \approx 3.9 \times 10^{-13}$.
- Thus η is a slow variable while the others are fast variables.
- We will use the fast and slow subsystems to understand the solution.

Reference: Christopher Jones, Geometric Singular Perturbation Theory.

Fast-Slow Hypotheses

We wish to use the theory of Fast-Slow Systems to solve for the invariant manifold.

- Notice we can write our system as

$$\begin{aligned} \dot{x} &= f(x, y, \epsilon) \\ \dot{y} &= \epsilon g(x, y, \epsilon) \end{aligned}$$

  where x are the fast variables and y are the slow variables.
- Notice that $f, g \in C^\infty$ since they are polynomials of the variables.
- Let M_0 be any compact subset of $\{(x, y)\,;\, f(x, y, \epsilon) = 0\}$.
  - Thus M_0 is a subset of $\{(x, y) : x = h^0(y)\}$ where $h^0(y)$ is defined for $y \in K$, a compact domain of $\mathbb{R}$.

Fast-Slow Overview

Given that the previous hypotheses are satisfied, Fenichel's Theorems assert that:

- There exists a manifold M_ϵ that lies within $\mathcal{O}(\epsilon)$ from M_0 and is diffeomorphic to M_0. Moreover it is locally invariant under the flow defined by our system.
- We can write $M_\epsilon = \{(x, y) : x = h^\epsilon(y)\}$ and thus we can write

$$\dot{y} = \epsilon g(h^\epsilon(y), y, \epsilon) = g(h^0(y), y, 0) + \mathcal{O}(\epsilon)$$

Fast Subsystem

- The Fast Subsystem is described by the system:

$$
\begin{aligned}
\dot{\eta} &= 0 \\
\dot{a} &= \frac{1}{R}Q\left(1-\frac{1}{2}\left(\alpha_s+\frac{1}{2}\left(\alpha_w+\alpha_i(\eta)\right)\right)\right)-A-(B+C)a+C\overline{T}, \\
\dot{z} &= \frac{1}{R}\left(Q(\alpha_i(\eta)-\alpha_w)-(B+C)z\right).
\end{aligned}
$$

- From this we can solve for the manifold M_0.
- Since this implies $\eta = constant$, we can easily solve for a and z on the manifold.

Slow Subsystem

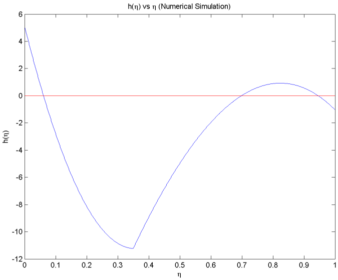
- Using the values from the manifold for a and z, we can solve

$$g(h^0(\eta),\eta,0) = a + \frac{1}{4}(e - z) + \frac{1}{2}(u_2 + d - s_2 z)p_2(\eta) - T_c.$$

- We can approximate the flow on the manifold M_ϵ by noting

$$\begin{aligned} \dot{\eta} &= \epsilon g(h^\epsilon(\eta),\eta,\epsilon), \\ &= g(h^0(\eta),\eta,0) + \mathcal{O}(\epsilon), \\ &= a + \frac{1}{4}(e - z) + \frac{1}{2}(u_2 + d - s_2 z)p_2(\eta) - T_c + \mathcal{O}(\epsilon). \end{aligned}$$

$h(\eta)$


h(η) vs η (Numerical Simulation)
h(η)
η
6
4
2
0
-2
-4
-6
-8
-10
-12
0
0.1
0.2
0.3
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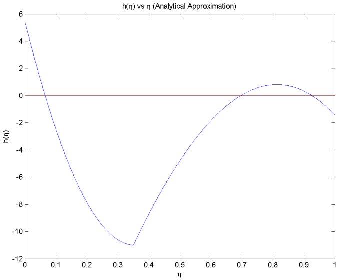
0.5
0.6
0.7
0.8
0.9
1


h(η) vs η (Analytical Approximation)
h(η)
η
6
4
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-2
-4
-6
-8
-10
-12
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
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1

Conclusion

- The scientific record is ambiguous between a Snowball Earth state and a Jormungand state.
- Taking into account the effect of the Hadley cells on equatorial ice-sheets we see the Budyko-Widiasih model gives a Jormungand state solution.
- We can receive an approximation to the solution using fast-slow theorems.

Further Analysis

Avenues of further analysis include:

- An investigation of the models against the climate record.
  - McGehee and Lehman analyzed the Budyko-Widiasih model against the climate record
  - Can do a comparative time series analysis
  - Specifically look at the regime switching or bifurcation.
- More investigations with GCMs.