Introduction to Continuous Dynamics
General Linear Behavior and the Linearization of Nonlinear Systems

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Abstract. In this paper we develop methods for analyzing the behavior of continuous dynamical systems near equilibrium points. We begin with a thorough analysis of linear systems and show that the behavior of such systems is completely determined by the eigenvalues of the matrix of coefficients. We then introduce the Stable Manifold and Hartman-Grobman theorems as a way to understand the dynamics of nonlinear systems near equilibrium points through a linearization of the system. The paper ends by showing how these theorems have been used in an actual application to solve for the dynamics near equilibrium points in climate models.

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CHAPTER 1

Introduction and the Linear Systems

Continuous dynamics is the study of the dynamics for systems defined by differential equations. These kinds of systems are commonly seen in areas such as Biology, Physics, and Climate Modeling. This paper will build a basis for the study of continuous dynamics with an introduction to the basic definitions, a study of the linear cases, and an introduction to the main theorems for understanding nonlinear systems through a linearized version of the system: the Hartman-Grobman theorem, and the Stable Manifold Theorem.

1.1. Basic Definitions

To start, we will define a system of differential equations.

DEFINITION. For $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, a **system of differential equations** is

$$X' = F(t, X) = \begin{pmatrix} f_1(t, x_1, \dots, x_n) \\ \vdots \\ f_n(t, x_1, \dots, x_n) \end{pmatrix}.$$

However, this definition is too general for the purpose of this paper. Instead we will look at autonomous systems.

DEFINITION. An **autonomous system of differential equations** is a system where every f_i has no explicit t dependence.

Note that if every f_i has no explicit t dependence, $F(t, X) = F(X)$. We will define a solution to the system as follows:

DEFINITION. A **solution** to the system, $X(t)$, satisfies the equation

$$X'(t) = F(X(t)).$$

This definition should make intuitive sense. It is says that the derivative of each x_i in our solution must be the same as if we plugged our solution $x_i(t)$ values into the f_i equations. If we take a solution define a starting point or initial value X_0, we then have the description of the dynamics of the object for all time. This is known as the flow which we formally define as:

DEFINITION. A function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is satisfying differential equation $X' = AX$ if with the initial value X_0 is called the **flow**. It is denoted by $\phi(t, X_0)$ and its output is coordinates of the object at time t that started at X_0.

We define an equilibrium point as follows:

DEFINITION. An **equilibrium point**, X_0, is a vector s.t. $F(X_0) = 0$.

Since we have an autonomous system, $F(X_0) = 0$ at a time t' means that the derivative of all of the system variables must always be zero for any $t \geq t'$. Thus in autonomous systems the object will stay at equilibrium points for the rest of time.

1.2. Introduction to Linear Systems

We will develop an understanding dynamical systems by understanding the dynamics around equilibrium points. Let's start by defining the different types of behavior for systems around equilibrium points using the simplest systems of differential equations: linear systems of differential equations.

DEFINITION. A **linear system** is a system of differential equations where each $f_i = a_{i_1}x_1 + \ldots + a_{i_n}x_n$.

A more intuitive version of this statement is that every f_i can be written as the linear combination of the variables. Note that we can define the matrix

$$A = \begin{pmatrix} a_{1_1} & \ldots & a_{1_n} \\ \vdots & & \vdots \\ a_{n_1} & \ldots & a_{n_n} \end{pmatrix}$$

and thus write $X' = AX$. We will continue writing linear systems in this form since linear algebra can lead us to powerful results using this formation. A useful concept is the exponential of a matrix. Recall from calculus

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This suggests the following definition:

Let A be an $n \times n$ matrix. We define the **exponential** of a matrix A to be

$$exp(A) = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

A proof that this sum converges for all $n \times n$ matrices is outside the scope of this paper [**2**]. The exponential is useful because it can be used to solve linear differential equations as seen in the following theorem:

THEOREM. *Let A be an $n \times n$ matrix. Then the unique solution of the initial value problem $X' = AX$ with $X(0) = X_0$ is $X(t) = e^{At}X_0$.*

The proof of this theorem is out the scope of this paper. Other results one can arrive at using the matrix formula involve the existence and uniqueness of equilibrium:

PROPOSITION. *If $\det A \neq 0$, then there exists a unique equilibrium point.*

This theorem follows because an equilibrium point is a vector X_0 where $AX_0 = 0$. From linear algebra we know that there is a unique solution to this system when $\det A \neq 0$. Tools from linear algebra also gives us the following proposition:

PROPOSITION. *If $\det A = 0$ then there exists a straight line of equilibrium points through the origin.*

Also important is that the tools of linear algebra provide us with another way of computing the solutions to a linear system with linearly independent eigenvectors:

THEOREM. *Suppose $V_1, \ldots, V_n$ are linearly independent eigenvectors for a matrix A and $\lambda_1, \ldots, \lambda_n$ are the respective eigenvalues. Then the solution to the system $X' = AX$ is $X(t) = \alpha_1 e^{\lambda_1 t}V_1 + \ldots + \alpha_n e^{\lambda_n t}V_n$ where $\alpha_1, \ldots, \alpha_n$ are constants.*

To prove this, simply write the matrix A in its eigenbasis $\{V_1, \ldots, V_n\}$ to obtain diagonalized matrix

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

Thus we can guess our solution to be

$$X(t) = \begin{pmatrix} \alpha_1 e^{\lambda_1 t} & & \\ & \ddots & \\ & & \alpha_n e^{\lambda_n t} \end{pmatrix}$$

as written in the eigenbasis for A. By taking the derivative of each equation $x_i(t) = \alpha_i \lambda_i e^{\lambda_i t}V_i$ we receive

$$X'(t) = \begin{pmatrix} \alpha_1 \lambda_1 e^{\lambda_1 t} & & \\ & \ddots & \\ & & \alpha_n \lambda_n e^{\lambda_n t} \end{pmatrix} = AX(t)$$

which completes our proof. This theorem is important because it gives the general solution to any linear system with linearly independent eigenvectors. The following theorem shows any matrix with n distinct eigenvalues have independent eigenvectors:

THEOREM. ***Independent Eigenvector Theorem.*** *If A is an $n \times n$ matrix with n distinct eigenvalues, then any set of n corresponding eigenvectors are independent.*

Assume there is some linearly dependent set of eigenvectors. Take the smallest subset of linearly dependent eigenvectors and denote its size as j. WLOG, denote this dependent set as $V_1, \ldots, V_j$. Since no eigenvector can be the zero vector, $j \geq 2$. Being linearly dependent implies that there are constants a_i s.t. at least one a_i is not zero and

$$a_1 V_1 + \ldots + a_j V_j = 0.$$

Assume WLOG $a_j \neq 0$. Thus writing $b_i = -\frac{a_i}{a_j}$ we can write the equation as

$$V_j = b_1 V_1 + \ldots + b_{j-1} V_{j-1}.$$

Since $V_j \neq 0$ we know that at least one b_i is nonzero. We can multiply this equation by A as written in its eigenbasis $\{V_1, \ldots, V_n\}$ to find

$$\lambda_j V_j = b_1 \lambda_1 V_1 + \ldots + b_{j-1} \lambda_{j-1} V_{j-1}$$

where λ_i is the eigenvalue associated with V_i. We can also add λ_j to the previous equation to get

$$\lambda_j V_j = b_1 z_j V_1 + \ldots + b_{j-1} z_j V_j$$

and then subtract this equation to the one just above to find

$$0 = b_1 (z_j - z_1) V_1 + \ldots + b_{j-1} (z_j - z_{j-1}) V_{j-1}$$

Thus we have found a set of $j - 1$ linearly dependent eigenvectors which gives a contradiction. Thus there is no linearly dependent set of eigenvectors and the proof is complete.

Now we want to be able to say that "most" or "almost all" matrices have n distinct eigenvalues. To do so, we can establish a concrete definition for what is meant by "most": we want a subset U to be "most" of its containing set if every point in the compliment of U can be approximated arbitrarily closely by points in U and no point in U can be approximated arbitrarily closely by points in the compliment of U. Thus U would have the property of being open and dense in its containing set. Noting $L(\mathbb{R}^n)$ is the set of $n \times n$ matrices, we define

Definition. P is a **generic property** if the set of all matrices having property P is open and dense in the set $L(\mathbb{R}^n)$.

Thus we can intuitively think that a generic property is a property that "almost all" matrices have. This leads us to our proposition:

Proposition. *Let P be the property that a matrix has n distinct eigenvalues. P is a generic property.*

The proof of this proposition is out the scope of this paper [**1**]. This result shows that we know the general solution to "almost all" $n \times n$ matrices. In addition, any matrix with repeating eigenvalues can be approximated arbitrarily closely by a matrix without repeating eigenvalues. Thus for the purposes of this paper we will only consider matrices without repeating eigenvalues. Note that this proposition implies that any matrix with repeating eigenvalues can be solved using a perturbation to a matrix without repeating eigenvalues to use the methods we describe.

1.3. Simple Planar Linear Systems

Planar linear systems are two-dimensional linear systems. Thus they can be written in the form:

$$\begin{aligned} x' &= ax + by, \\ y' &= cx + dy. \end{aligned}$$

We will consider only such equations where the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct eigenvalues. From the previous section we know that the general solution to such equations is $X(t) = \alpha_1 e^{\lambda_1 t} V_1 + \alpha_2 e^{\lambda_2 t} V_2$ where V_1, V_2 are the eigenvectors of A and λ_1,λ_2 are the respective eigenvalues. We will see that the solution to such a system is completely determined by the eigenvalues and eigenvectors. We can understand the complete behavior of such systems by investigating the different possibilities for the eigenvalues and eigenvectors. To start, let's examine the cases where $V_1 = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $V_2 = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

1.3.1. Source. Let's begin by looking at the case where both eigenvalues are positive. This case is known as a **source**. Notice that when $\lambda > 0$, as $t \to \infty$, $|\alpha e^{\lambda t}| \to \infty$ for any $\alpha \in \mathbb{R}$. Also, the term with the eigenvalue of greater magnitude grows faster. More concretely, if $|\lambda_1| > |\lambda_2|$, then $\lambda_1 > \lambda_2$ and thus

$$\lim_{t\to\infty} \frac{\alpha_2 e^{\lambda_2 t}}{\alpha_1 e^{\lambda_1 t}} = \lim_{t\to\infty} \frac{\alpha_2}{\alpha_1} e^{(\lambda_2-\lambda_1)t} = 0.$$

Thus when $|\lambda_1| > |\lambda_2|$ we can say that when $\alpha_1 \neq 0$

$$\lim_{t\to\infty} X(t) \approx \alpha_1 e^{\lambda_1 t} V_1.$$

From this we can conclude

$$\lim_{t\to\infty} X(t) = \text{sign}(\alpha_1)\infty$$

where

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

From these results we receive the general solution depicted in Figure 1.3.1. This picture is called a **phase portrait** of the system. Representative flows are depicted with arrows showing the direction that the solution would move during a forward progression of time. Here we see all of the flows as going away from the equilibrium point (the origin). As expected, for every $X_0 \in \mathbb{R}^2$, $|\phi(t, X_0)| \to \infty$ as $t \to \infty$. Notice that in this picture we depict $\lambda_1 > \lambda_2$ and thus when α_1 is positive (the values of the flow have a positive x-coordinate), then $X(t)$ approaches ∞ while when a_2 is negative we have $X(t)$ approach $-\infty$. Notice that the x and y axis depict cases where one of the α_i values is 0 and thus the solution never leaves the respective axis.

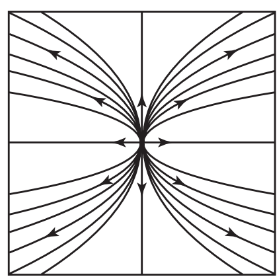

FIGURE 1.3.1. Source. The eigenvalue for the e_1 has a greater magnitude.

1.3.2. Sink. Now take the case where both eigenvalues are negative. This case is known as a **sink**. Notice that when $\lambda < 0$, as $t \to \infty$, $\alpha e^{\lambda t} \to 0$ for any $\alpha \in \mathbb{R}$. Thus

$$\lim_{t\to\infty} X(t) = 0.$$

Notice that the eigenvalue with the greater magnitude shrinks faster. Formally, if $|\lambda_1| > |\lambda_2|$, then $\lambda_1 < \lambda_2$ and thus

$$\lim_{t\to\infty} \frac{\alpha_2 e^{\lambda_2 t}}{\alpha_1 e^{\lambda_1 t}} = \lim_{t\to\infty} \frac{\alpha_2}{\alpha_1} e^{(\lambda_2-\lambda_1)t} = 0.$$

Thus when $|\lambda_1| > |\lambda_2|$ we can say that when $\alpha_2 \neq 0$

$$\lim_{t\to\infty} X(t) \approx \alpha_2 e^{\lambda_2 t} V_2.$$

From these results we can receive the general solution depicted in Figure 1.3.2. All of the flows going towards the origin as expected. In the limit, most of the flows seem to fall towards the origin from the y-axis. The only flows that do not approach the origin from the y-axis are those on the x-axis which correspond to points where $\alpha_2 = 0$.

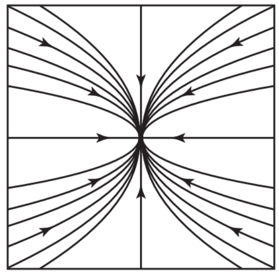

FIGURE 1.3.2. Sink. The eigenvalue for the e_1 has a greater magnitude.

1.3.3. Saddle. We can couple the ideas from the previous two cases to understand the case known as the **saddle**. A saddle is the case where one eigenvalue is positive and the other is negative. If we assume $\lambda_1 < 0 < \lambda_2$, we see that as $t \to \infty$, $\alpha_1 e^{\lambda_1 t} \to 0$ and $\alpha_2 e^{\lambda_2 t} \to \text{sign}(\alpha_2)\infty$. Putting these facts together we notice

$$\lim_{t\to\infty} X(t) \approx \alpha_2 e^{\lambda_2 t} V_2$$

and thus when $\alpha_2 \neq 0$

$$\lim_{t\to\infty} X(t) = \text{sign}(\alpha_2)\infty.$$

An example of such a system is depicted in Figure 1.3.3. We see that in the limit all of the flows go towards positive or negative infinity (respective of the sign of α_2) and approach the y-axis. However, when $\alpha_2 = 0$ we have a solution which is written as $\alpha_1 e^{\lambda_1 t}$ and thus heads towards the origin.

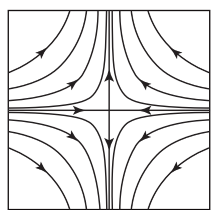

FIGURE 1.3.3. Saddle. The e_1 basis has the negative eigenvalue

1.3.4. Center. There is a case with periodic orbits known as a center. A **center** is defined by a matrix $A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$ where $\beta \neq 0$. Such a matrix gives rise to eigenvalues $\pm i\beta$. We solve for the eigenvector for $\lambda = i\beta$ by noticing

$$\begin{pmatrix} -i\beta & \beta \\ -\beta & -i\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies $i\beta x = \beta y$ or $V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Thus

$$X(t) = e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

By Euler's Formula

$$e^{i\beta t} = \cos\beta t + i \sin\beta t$$

and thus

$$X(t) = \begin{pmatrix} \cos\beta t + i\sin\beta t \\ i(\cos\beta t + i\sin\beta t) \end{pmatrix} = \begin{pmatrix} \cos\beta t + i\sin\beta t \\ -\sin\beta t + i\cos\beta t \end{pmatrix}.$$

If we write $X(t)$ in its real and imaginary parts

$$X(t) = X_{Re}(t) + iX_{Im}(t)$$

we see

$$X'(t) = X'_{Re}(t) + iX'_{Im}(t) = AX(t) = AX_{Re}(t) + iAX_{Im}(t).$$

By equating the real parts and equating the imaginary parts we see both $X_{Re}(t)$ and $X_{Im}(t)$ are solutions. But also notice

$$X_{Re}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; X_{Im}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and thus the general solution is $X(t) = \alpha_1 X_{Re}(t) + \alpha_2 X_{Im}(t)$. Notice that each solution is periodic with period $\frac{2\pi}{\beta}$. Thus the solution is depicted in Figure 1.3.4.

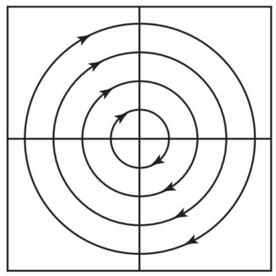

FIGURE 1.3.4. Center

1.3.5. Spiral Source/Sink. Our last simple case is a spiral. It is represented by the matrix $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ where $\alpha, \beta \neq 0$. Such a matrix gives rise to eigenvalues $\alpha \pm i\beta$. Following the derivation of the center we arrive

$$X(t) = e^{(\alpha + i\beta)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and thus receive the general solution

$$X(t) = \alpha_1 e^{\alpha t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + \alpha_2 e^{\alpha t} \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}$$

which is the same as the solution for the center except for the factor $e^{\alpha t}$. Notice that this implies that if $\alpha > 0$ then as $t \to \infty$, $X(t) \to \infty$ and likewise if $\alpha < 0$ then as $t \to \infty$, $X(t) \to \infty$. Thus our periodic solution is multiplied by a factor that expands or contracts space according to α leading to a spiral shape. We call the case where $\alpha > 0$ a **spiral source** as the solution spirals outward and the case where $\alpha < 0$ is a **spiral sink**. Example solutions are depicted in Figure 1.3.5.

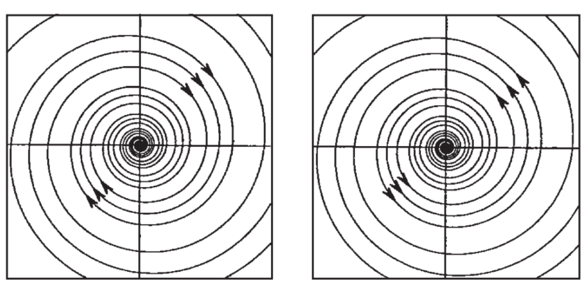

FIGURE 1.3.5. The left is a spiral source and the right is a spiral sink

1.4. General Linear Behavior

We can use our knowledge of the simple cases of planar linear dynamics to generalize to all linear cases. We will start by describing the rest of the planar linear solutions and describe how one would generalize the analysis to higher dimensions.

1.4.1. Linear Transformations and Coordinate Changes. First we wish to address the problem of matrices with an eigenbasis other than $\{e_1, e_2\}$. To understand matrices whose eigenbasis is not $\{e_1, e_2\}$, note that we can always change our coordinates to receive matrices whose eigenbasis are $\{e_1, e_2\}$ in the new coordinates thus understand the system in the new coordinates. Thus we define:

DEFINITION. A **linear map** or **linear transformation** is a function $T : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

THEOREM. *Suppose the matrix A has n real, distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with associated eigenvectors $V_1, \dots, V_n$. Let T be the matrix whose columns are $V_1, \dots, V_n$. Thus*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

This theorem shows that for every matrix with real eigenvalues we can find a coordinate change from $\{e_1, e_2\}$ to the eigenbasis of A $\{V_1, V_2\}$ and thus understand any system with real eigenvalues through this linear transformation. Figure 1.4.1 provides an example of such a shifted sink. Notice that the analysis for a center and a spiral above did not rely on an eigenbasis of $\{e_1, e_2\}$ and thus this theorem will show that we understand the solutions for all planar linear systems. Note that this also implies that the same idea will be able to be applied to higher dimensional systems as well.

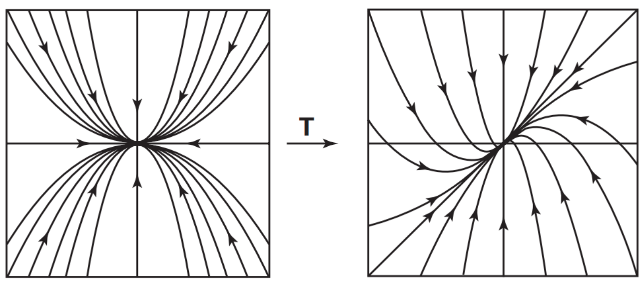

FIGURE 1.4.1. Linear transformation of a sink to a new basis

The proof is as follows. By the construction of T, $Te_j = V_j$ for $j = 1, \dots, n$. From the Independent Eigenvector Theorem we know the columns of T are linearly independent and thus $\det T \neq 0$. Thus T^{-1} exists thus $T^{-1}V_j = e_j$. Therefore we see

$$(T^{-1}AT)e_j = T^{-1}AV_j = T^{-1}(\lambda_j V_j) = \lambda_j T^{-1}V_j = \lambda_j e_j$$

and thus $\{e_1, \dots, e_n\}$ is the eigenbasis for $(T^{-1}AT)$ with eigenvalues $\lambda_1, \dots, \lambda_n$ proving the theorem.

1.4.2. Conjugacies of Hyperbolic Matrices. We will start by understanding the relationship between the dynamics of all solutions whose eigenbasis is $\{e_1, e_2\}$. To do this we will establish a relationship between the long-term dynamics of systems described above. Thus we will define our relationship:

DEFINITION. Suppose $X' = AX$ and $X' = BX$ have the respective flows ϕ_A and ϕ_B. These two systems are **topologically conjugate** if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$that satisfies

$$\phi_B(t, h(X_0) = h(\phi_A(t, X_0)).$$

Thus notice that a topological conjugacy establishes a one-to-one, onto, continuous, and inverse continuous mapping from flows of $X' = AX$ to flows of $X' = BX$. Therefore we will say that A and B are **dynamically equivalent** if A is topologically conjugate to B. Define the following term:

DEFINITION. A matrix A is **hyperbolic** if none of its eigenvalues has real part 0. The system $X' = AX$ is also called hyperbolic.

Using this definition we can assert and prove our theorem:

THEOREM. *Suppose that A_1 and A_2 are 2×2 matrices and are both hyperbolic. Then the linear systems $X' = A_i X$ are conjugate if and only if each matrix has the same number of eigenvalues with negative real part.*

First we will establish that any two matrices which differ by a linear transformation are topologically conjugate.

LEMMA. *Suppose $X' = A_1 X$ and $X' = A_2 X$ has the same eigenvalues λ_1 and λ_2. A_1 is topologically conjugate to A_2.*

Assume WLOG A_1 has the eigenbasis $\{e_1, e_2\}$ and denote the eigenbasis of A_2 as V_1,V_2.. Thus there exists a linear transformation T from A_1 to A_2. We know from an earlier theorem that T maps e_1 to V_1 and e_2 to V_2. Thus let $h(X_0) = TX_0$ and write

$$X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Thus we can write

$$X = \begin{pmatrix} x \\ y \end{pmatrix}_{V_1, V_2}$$

to be the vector as written in the V_1, V_2 basis we see

$$h(\phi_{A_1}(t, X_0)) = h(\begin{pmatrix} x_0 e^{\lambda_1 t} \\ y_0 e^{\lambda_2 t} \end{pmatrix}) = T \begin{pmatrix} x_0 e^{\lambda_1 t} \\ y_0 e^{\lambda_2 t} \end{pmatrix} = \begin{pmatrix} x_0 e^{\lambda_1 t} \\ y_0 e^{\lambda_2 t} \end{pmatrix}_{V_1, V_2} = \phi_{A_2}(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}_{V_1, V_2}) = \phi_{A_2}(t, h(X_0)).$$

Therefore A_1 is topologically conjugate to A_2. Notice that this implies that any two matrices with the same eigenvalues are topologically conjugate.

We also need a fact from linear algebra:

PROPOSITION. *Every matrix can be linear transformed into a matrix of the Jordan canonical form. These are the matrices*

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Thus, given our lemma, we only need to prove the theorem for the cases where $A_1$and $A_2$are in one of the Jordan canonical forms.

Case 1. Suppose both linear systems $X' = A_i X$ have two nonzero real eigenvalues λ_i and μ_i where the λ_i's have the same sign and the μ_i's have the same sign. Assume WLOG that the eigenbasis for both matrices is $\{e_1, e_2\}$. Thus we have matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Notice we can assume our matrices have the eigenbasis $\{e_1, e_2\}$ because our lemma shows that A_1 and A_2 are topologically conjugate to such a matrix. Thus we have the differential equations $x' = \lambda_i x$ and $y' = \lambda_i y$. Define

$$h_1(x) = \begin{cases} x^{\frac{\lambda_2}{\lambda_1}} & \text{if } x \geq 0 \\ -|x|^{\frac{\lambda_2}{\lambda_1}} & \text{if } x < 0 \end{cases},$$

$$h_2(y) = \begin{cases} y^{\frac{\mu_2}{\mu_1}} & \text{if } y \geq 0 \\ -|y|^{\frac{\mu_2}{\mu_1}} & \text{if } y < 0 \end{cases},$$

to construct $H(x, y) = (h_1(x), h_2(y))$. Notice that the reason we must use such a function h is that if we use $h_1(x) = x^{\frac{\lambda_2}{\lambda_1}}$, we can have cases such as $\lambda_2 = 2$ and $\lambda_1 = 1$ where h is not injective. Also we have to require that the eigenvalues have the same sign to ensure continuity. Thus if we look at the x coordinate we see that when $x \geq 0$,

$$h(\phi_{A_1}(t, x_0)) = h(x_0 e^{\lambda_1 t}) = (x_0 e^{\lambda_1 t})^{\frac{\lambda_2}{\lambda_1}} = x_0^{\frac{\lambda_2}{\lambda_1}} e^{\lambda_2 t} = \phi_{A_2}(t, x_0^{\frac{\lambda_2}{\lambda_1}}) = \phi_{A_2}(t, h(x_0)).$$

An almost identical computation shows the same is true when $x < 0$. We see the same computations hold for the y coordinate. Thus A_1 is topologically conjugate to A_2.

Case 2. Suppose the linear system $X' = AX$ has a matrix that takes the form

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}.$$

Assume WLOG $\alpha < 0$. We wish to show that the the system is conjugate to $X' = BX$ where

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\tau = \tau(X_0)$ be the time such that $|\phi_A(\tau, X_0)| = 1$, the time that the solution with the initial value X_0 intersects the unit circle S^1. Notice that, given B has two eigenvalues of -1,

$$\phi_B(t, X) = e^{-t}X.$$

Define the conjugacy

$$H(X) = \phi_B(-\tau, \phi_A(\tau, X)))$$

and let $H(\vec{0}) = \vec{0}$. To show this is the conjugacy we want, first notice

$$\phi(t_1 + t_2, X) = \phi(t_1, \phi(t_2, X))$$

which implies

$$\phi_A(\tau - s, \phi_A(s, X_0)) = \phi_A(\tau, X_0) \in S^1$$

and thus

$$\tau(\phi_A(s, X_0)) = \tau - s.$$

Therefore

$$\begin{aligned} H(\phi_A(s, X_0)) &= \phi_B(-\tau + s, \phi_A(\tau - s, \phi_A(s, X_0))), \\ &= \phi_B(s, \phi_B(-\tau, \phi_A(, \tau, X_0))), \\ &= \phi_B(s, H(X_0)). \end{aligned}$$

In order for this to be a conjugacy we must also show H is a homeomorphism. We define the function

$$G(X) = \phi_A(-\tau_1, \phi_B(\tau_1, X))$$

and set $G(\vec{0}) = \vec{0}$ where $\tau_1 = \tau_1(X_0)$ is the time such that $|\phi_B(\tau_1, X_0)| = 1$. It is clear from the construction that $G^{-1} = H$ and thus H is bijective. Notice that $\phi_B(t, X) = e^{-t}X$ which implies $\tau_1 = \log r$ where $r^2 = x^2 + y^2$. Thus

$$\phi_B(\tau_1, X) = \phi_B(\log r, X) = e^{-\log r}X = \frac{1}{r}X$$

which means we can write G equivalently as

$$\phi_A(-\log r, \frac{1}{r}X)$$

and since flows are continuous, we know H^{-1} is continuous at all points except $X_0 = \vec{0}$. To show G is continuous at the origin, take X_0 close to the origin. Recall that as $r \to 0$, $-\log r \to \infty$. Thus $\phi_A(-\log r, \frac{1}{r}X) \approx \vec{0}$. Since $\frac{1}{r}X \in S^1$, the filled unit circle is mapped close to the origin and thus G is continuous at the origin. Now we must show H is continuous. Since flows are continuous, we simply need to show $\tau(X)$ is continuous. Notice

$$\frac{\partial}{\partial t}|\phi_A(t, X)| = \frac{\partial}{\partial t}\sqrt{x(t)^2 + y(t)^2} = \frac{x(t)x'(t) + y(t)y'(t)}{|\phi_A(t, X)|}$$

and when $t = \tau(X)$ we know that the vector field $(x'(t), y'(t))$ points inside of S^1 and thus $(x'(t), y'(t)) \neq (0, 0)$ which implies

$$\frac{\partial}{\partial t}|\phi_A(t, X)| \neq 0$$

at $(\tau(x, y), x, y)$. Thus we can apply the implicit function theorem to solve for a differentiable function for τ which implies τ is differentiable and thus τ is continuous giving us continuity in H [**6**]. The argument that H is continuous at $(0, 0)$ closely follows the argument that H^{-1} is continuous at $(0, 0)$. Thus H is a homeomorphism and thus a conjugacy between $X' = AX$ and $X' = BX$.

Notice that the only part of this case that requires $\alpha < 0$ is part proving G is continuous at $(0, 0)$. However, if $\alpha > 0$ then we can define the conjugacy as

$$H(X) = \phi_B(\tau, \phi_A(-\tau, X)))$$

and the proof will follow.

Case 3. Suppose the linear system $X' = AX$ has a matrix of the form

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}$$

where $\epsilon > 0$ sufficiently small. Thus

$$T^{-1}AT = \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda \end{pmatrix}$$

and thus we can continue to prove our theorem using the argument in Case 2.

1.4.3. Higher Dimensional Systems. All of the theorems and definitions above generalize to higher dimensions. The solution type is completely determined by the eigenvalues of the matrix and their behavior on the eigenvector is the same as the analysis above. The only difference is there are more axes to analyze. An example is shown in Figure 1.4.2. This figure is known as a spiral saddle. The eigenvectors are e_1, e_2, and e_3. The eigenvalues for the e_1 and e_2 eigenvectors are $\alpha \pm i\beta$ where $\alpha < 0$ as seen by the spiral sink in the xy-plane. The eigenvalue for the e_3 eigenvector is positive which shows the repulsion of the solution on the e_3 axis.

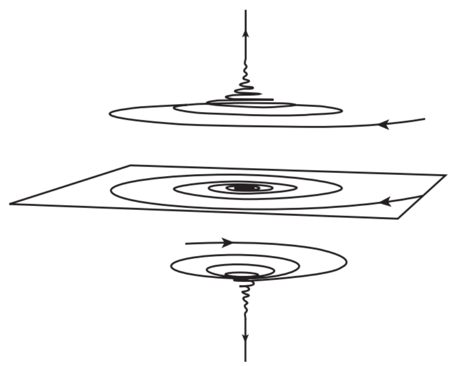

FIGURE 1.4.2. Spiral Saddle

Also note that it can be proved for the hyperbolic $n \times n$ matrices A_1 and A_2, the linear systems $X' = A_iX$ are conjugate if and only if each matrix has the same number of eigenvalues with negative real part.

CHAPTER 2

Nonlinear Systems

Nonlinear systems exhibit more complexity than linear systems. In general, there is no guarantee that there in the existence and uniqueness of an equilibrium point. The solutions to a system may not only be linear combinations of elements who themselves are solutions. Lastly, these systems are not in a form that allows us to place the coefficients in a matrix and use the tools from linear algebra to arrive at theorems. Thus in many cases nonlinear systems are much harder to analyze.

However, there are tools for understanding nonlinear systems. The goal of this chapter will be to introduce some the most important tools for understanding nonlinear systems by understanding related linear systems. The main theorems of this topic are the the Stable Manifold Theorem and the Hartman-Grobman Theorem.

2.1. The Stable Manifold Theorem

The Stable Manifold Theorem in its intuitive form states that the set of points that an equilibrium point attracts is well approximated by a linear system near the equilibrium point (and the same holds for the set of points repelled) [**4, 7**]. This linearization of a system is what is known as the total derivative. The total derivative finds a linear tangent hypersurface near the equilibrium point which can serve as a good approximation for some arbitrarily small neighborhood around the equilibrium point. So to start, we define the total derivative as follows:

Definition. The function $F : \mathbb{R}^n \to \mathbb{R}^n$ is **differentiable** at $X_0 \in \mathbb{R}^n$ if there is a linear transformation $DF(X_0) \in L(\mathbb{R}^n)$ that satisfies

$$\lim_{|H|\to 0} \frac{|F(X_0 + H) - F(X_0) - DF(X_0)H|}{|H|} = 0.$$

The linear transformation $DF(X_0)$ is called the **total derivative of** F **at** X_0. The linear system of differential equations

$$X' = DF(X_0)X$$

is called the **linearized system near** X_0**.** If $DF(X_0)$ is a hyperbolic matrix, then we say X_0 is a **hyperbolic equilibrium point.**

By using this definition of a derivative in conjunction with Taylor's Theorem we see

$$F(X) = DF(\vec{0})X + \frac{1}{2}D^2F(\vec{0})(X, X) + \dots$$

which implies $DF(\vec{0})$ is a good first approximation to $F(X)$ near $X = \vec{0}$. Given these definition, we can prove the following theorem:

Theorem. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable at X_0, then the partial derivative $\frac{\partial f_i}{\partial x_j}$, $i, j = 1, \dots, n$ all exist at X_0 and for all $x \in \mathbb{R}^n$,*

$$DF(X_0)X = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(x_0)x_j$$

and thus if F is a differentiable function, the derivative DF is given by the $n \times n$ Jacobian matrix

$$Df = \left(\frac{\partial f_i}{\partial x_j}\right)$$

The proof of this theorem is out the scope of the paper. For the purposes of this paper, we will constrain our focus to nonlinear systems which are continuously differentiable. These are systems which have a continuous derivative and are denoted by the set C^1. We denote that a function F is continuously differentiable on a subset E of $\mathbb{R}^n$ as $C^1(E)$. We can prove the following about the existence and uniqueness of equilibrium for C^1 functions:

THEOREM. *Consider the initial value problem*

$$X' = F(X),\ X(0) = X_0$$

where $X_0 \in \mathbb{R}^n$. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then there exists a unique solution to this differential equation satisfying the initial condition $X(t_0) = X_0$.

The proof of this theorem is out the scope of this paper. We also need the following definitions:

The **Stable Manifold of an equilibrium point** X_0 is

$$E_s = \{X \in \mathbb{R}^n : \lim_{t\to\infty} \phi(t, X) = X_0\}.$$

and the **Unstable Manifold of** X_0 **is**

$$E_U = \{X \in \mathbb{R}^n : \lim_{t\to-\infty} \phi(t, X) = X_0\}.$$

An intuitive way to understand these definitions is that the stable manifold of an equilibrium point is the set of all of the points who are attracted to the equilibrium while the unstable manifold is the set of all points which are repelled from the equilibrium. These manifolds in the linear case were simple: the space spanned by the eigenvectors with positive real part eigenvalues were all repelled and thus formed the unstable manifold of the equilibrium point. The space spanned by the eigenvectors with negative real part eigenvalues were all attracted and thus formed the stable manifold of the equilibrium point.

What we wish to show is that the linearized system near the equilibrium gives a good approximation to the stable and unstable manifolds. Just as in calculus, the "good approximation" to the hypersurface at a given point is the tangent hypersurface. Thus what we wish to show is that the stable and unstable manifolds for the linearized version are tangent to the manifolds of the nonlinear version at the equilibrium point.

THEOREM. ***The Stable Manifold Theorem.*** *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $F \in C^1$ and let $\phi(t, X)$ be the flow of the nonlinear system $X' = F(X)$. Let $F(\vec{0}) = \vec{0}$ and let $DF(\vec{0})$ have k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable manifold E_s of the linearized system at $\vec{0}$ such that for all $t \geq 0$, $\phi(t, S) \subset S$ and for all $X_0 \in S$,*

$$\lim_{t\to\infty} \phi(t, X_0) = \vec{0}.$$

Also, there exists an $n-k$ dimensional differentiable manifold U tangent to the unstable manifold E_U of the linearized system at $\vec{0}$ such that for all $t \leq 0$, $\phi(t, U) \subset U$ and for all $X_0 \in U$,

$$\lim_{t\to-\infty} \phi(t, X_0) = \vec{0}.$$

Note that we can always change the coordinates of any problem such that the equilibrium point is at the origin as required by this statement of the theorem. Given that $F(\vec{0}) = \vec{0}$, we can write

$$X' = AX + \bar{F}(X)$$

where $A = DF(\vec{0})$, $G(X) = F(X) - AX$, $F \in C^1(E)$, $DF(\vec{0}) = \vec{0}$. Because our function is differentiable at X_0, for every $\epsilon > 0$, there is a $\delta > 0$ such that $|X - Y| \leq \delta$ implies

$$|F(X) - F(Y)| \leq \epsilon |X - Y|.$$

One could think of this as following from our definition of the derivative as it requires the multidimensional slope to limit to $\vec{0}$ (though this can be proven through a more rigorous method). From linear algebra we know that can always find a linear transformation L to write A in the form

$$B = L^{-1}AL = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

where the eigenvalues $\lambda_1, \ldots, \lambda_k$ of the $k \times k$ matrix P have negative real part and the eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ of the $(n - k) \times (n - k)$ matrix Q have positive real part. Choose $\alpha > 0$ sufficiently small such that for $j = 1, \ldots, k$,

$$Re(\lambda_j) < -\alpha < 0$$

and thus by letting $Y = L^{-1}X$ we can write the system in the form

$$Y' = BY + G(Y)$$

where $G(y) = L^{-1}\bar{F}(LY) \in C^1(\tilde{E})$ where $\tilde{E} = L^{-1}E$. Let

$$U(t) = \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix}, V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix}.$$

Thus $U' = BU$ and $V' = BV$ which in turn gives

$$e^{Bt} = U(t) + V(t).$$

Since we have chosen a sufficiently small $-\alpha > Re(\lambda_j)$ for all of the negative eigenvalues, we can choose a $\sigma > 0$ sufficiently small and a $K > 0$ sufficiently large such that

$$||U(t)|| \leq Ke^{-(\alpha+\sigma)t}$$

for all $t \geq 0$ and

$$||V(t)|| \leq Ke^{\sigma t}$$

for all $t \leq 0$. Now consider the equation

$$U(t,a) = U(t)a + \int_0^t U(t-s)G(U,(s,a)))ds - \int_t^\infty V(t-s)G(U(s,a))ds$$

Notice that $U(t,a)$ is the solution to $Y' = BY + G(Y)$. This integral can be solved using the method of successive approximation to show

$$U_1(t,a) = 0$$

$$|U_j(t,a) - U_{j-1}| \leq \frac{K|a|e^{-\alpha t}}{2^{j-1}}$$

where

$$\lim_{j\to\infty} U_j(t,a) = U(t,a)$$

which gives that

$$|U(t,a)| \leq 2K|a|e^{-\alpha t}$$

when $|a|$ is sufficiently small. By looking at the integral equation we see that the last $n-k$ components of a do not enter the computation and thus can be taken to be zero. Thus the components of $U(t,a)$ must satisfy the initial conditions

$$(U(t,a))_j = a_j \text{ for } j = 1,\ldots,k.$$

We then define the function

$$\psi(\psi_{k+1}(a),\ldots,\psi_n(a))$$

where

$$\psi_j(a) = (U(0,(a_1,\ldots,a_k,0,\ldots,0))_j.$$

From here it can be shown that by setting $Y(\vec{0}) = U(0,a)$,

$$Y(t) = U(t,a)$$

and that

$$\frac{\partial \psi_j}{\partial y_i}(\vec{0}) = \vec{0}$$

and thus the differentiable manifold S is tangent to the stable subspace $E^s = \{y \in \mathbb{R}^n : y_1 = \ldots = y_k = 0\}$ [**7**]. A similar proof shows the analogous conclusion for the unstable subspace.

2.2. The Hartman-Grobman Theorem

The Hartman-Grobman Theorem, also known as the Linearization theorem, develops the similarity between the nonlinear system and the linearized version near an equilibrium point by establishing a topological conjugacy. From the Stable Manifold Theorem we know that the stable and unstable manifolds are well approximated by the linearized system near the equilibrium point. The Hartman-Grobman Theorem goes the next step to establish that the long-term behavior of the two systems near the equilibrium point are similar as denoted by a topological conjugacy.

THEOREM. *Take $X \in \mathbb{R}^n$ and consider the nonlinear system $X' = F(X)$ with the flow $\phi(t, X)$ and the linear system $X' = AX$ where $A = DF(X^*)$ and X^* is a hyperbolic fixed point. Assume we have translated X^* such that $X^* = \vec{0}$.*

Let f be C^1 on some $E \subset \mathbb{R}^n$ with $\vec{0} \in E$. Let $I_0 \subset \mathbb{R}$, $U \subset \mathbb{R}^n$, and $V \subset \mathbb{R}^n$ such that U, V, and I_0 all contain the origin. Then there exists a homeomorphism $H : U \to V$ such that, for all $\vec{X_0} \in U$ and all $t \in I_0$,

$$H(\phi(t, X_0)) = e^{At} H(X_0)$$

and thus $X' = F(X)$ is topologically conjugate to $X' = AX$.

The proof is outside the scope of this paper. However the proof follows a structure similar to the Stable Manifold Theorem proof [**9**, **2**]. One starts by supposing the matrix can be written in the form

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

which separates the positive real eigenvalues in the matrix Q from the negative real eigenvalues in the matrix P. From here one establishes a sufficiently small neighborhood around 0 and develops the homeomorphism H using the method of successive approximations. Then one shows that the equation

$$H(\phi(t, X_0)) = e^{At} H(X_0)$$

is sufficiently satisfied in the neighborhood around X_0 which completes the proof.

2.3. Application of the Hartman-Grobman Theorem

The Hartman-Grobman Theorem can easily be applied to nonlinear systems in order to understand the behavior of the solution near equilibrium points. An example comes from a paper written by Dr. Richard McGehee and Dr. Esther Widiasih [**5**]. These researchers develop a nonlinear dynamical system to describe the ice-albedo feedback system. After much simplification they notice that the system can be written as

$$\dot{\eta} = \epsilon\left(w + \frac{Qs_2(1-\alpha_0)}{B+C} p_2(\eta) - T_c\right),$$

$$\dot{w} = \frac{1}{R}\left(B\Phi_0(\eta) - Bw - \epsilon\Omega(w + \frac{Qs_2(1-\alpha_0)}{B+C} p_2(\eta) - T_c\right).$$

Thus the equilibrium points can be found by solving

$$h(\eta) = \Psi_0(\eta) + \frac{Qs_2(1-\alpha_0)}{B+C} p_2(\eta) - T_c = 0.$$

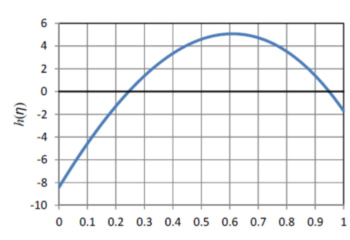

FIGURE 2.3.1. The graph of $h(\eta)$

From Figure 2.3.1 we can see that $h(\eta)$ has two such solutions and thus there are two equilibrium points, $\eta_1 \approx .25$ and $\eta_2 \approx .95$. Thus to understand the behavior near the equilibrium points, we can find the Jacobian matrix for the system at the equilibrium points

$$\begin{pmatrix} \frac{\partial \dot{\eta}}{\partial \eta} & \frac{\partial \dot{\eta}}{\partial w} \\ \frac{\partial \dot{w}}{\partial \eta} & \frac{\partial \dot{w}}{\partial w} \end{pmatrix} = \begin{pmatrix} \epsilon \frac{Qs_2(1-\alpha_0)}{B+C} p_2'(\eta_i) & \epsilon \\ \frac{B}{R}\Phi_0'(\eta_i) - \frac{\epsilon\Omega}{R}\frac{Qs_2(1-\alpha_0)}{B+C} p_2'(\eta_i) & -\frac{B}{R} \end{pmatrix}$$

which has eigenvalues approximated by $-\frac{B}{R}$ and$\epsilon h'(\eta_i)$. Thus for $\eta = .25$, the Jacobian has one positive eigenvalue and one negative eigenvalue making η_1 a saddle point. For $\eta = .95$, the Jacobian has two negative eigenvalues making η_2 a sink.

CHAPTER 3

Conclusion

The purpose of this paper was to develop methods for understanding the dynamics of continuous dynamical systems using linearization methods. In this paper we have developed a thorough understanding of the possible solutions to linear systems of differential equations. Our method involved solving for the eigenvalues of the matrix of coefficients which completely determined the long-term behavior. We then extended this method to nonlinear systems through the Stable Manifold and the Hartman-Grobman theorems. These theorems show how a nonlinear system can be analyzed near the equilibrium points through a linearized form of the system known as the Jacobian. With this information we were able to understand how climate researchers were able to conclude that certain equilibrium points in their nonlinear model were saddles and sinks.

There were portions of this paper which could have been exanded on, but due to the lenghth of the paper were not delved in to. Most were theorems of linear algebra such as the fact that the summation that defines the exponential of a matrix converges for all $n \times n$ matrices. For rigorous treatment of such topics, please consult the books *Differential Equations, Dynamical Systems, and in Introduction to Chaos* by Hirsch et. all and *Differential Equations and Dynamical Systems* by Perko Lawrence [**1, 2**].

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